

# THE UNIVERSAL K3 SURFACE OF GENUS 14 VIA CUBIC FOURFOLDS

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**ABSTRACT.** Using Hassett's isomorphism between the Noether-Lefschetz moduli space  $\mathcal{C}_{26}$  of special cubic fourfolds  $X \subset \mathbf{P}^5$  of discriminant 26 and the moduli space  $\mathcal{F}_{14}$  of polarized  $K3$  surfaces of genus 14, we use the family of 3-nodal scrolls of degree seven in  $X$  to show that the universal  $K3$  surface over  $\mathcal{F}_{14}$  is rational.

## 1. INTRODUCTION

For a very general cubic fourfold  $X \subseteq \mathbf{P}^5$ , the lattice  $A(X) := H^{2,2}(X) \cap H^4(X, \mathbb{Z})$  of middle Hodge classes contains only classes of complete intersection surfaces, so  $A(X) = \langle h^2 \rangle$ , where  $h \in \text{Pic}(X)$  is the hyperplane class (see [V]). Hassett, in his influential paper [H1], initiated the study of Noether-Lefschetz special cubic fourfolds. If  $\mathcal{C}$  is the 20-dimensional coarse moduli space of smooth cubic fourfolds  $X \subseteq \mathbf{P}^5$ , let  $\mathcal{C}_d$  be the locus of *special* cubic fourfolds  $X$  characterized by the existence of an embedding of a saturated rank 2 lattice

$$L := \langle h^2, [S] \rangle \hookrightarrow A(X),$$

of discriminant  $\text{disc}(L) = d$ , where  $S \subseteq X$  is an algebraic surface not homologous to a complete intersection. Hassett [H1] showed that  $\mathcal{C}_d \subseteq \mathcal{C}$  is an irreducible divisor, which is nonempty if and only if  $d > 6$  and  $d \equiv 0, 2 \pmod{6}$ . The study of the divisors  $\mathcal{C}_d$  for small  $d$  has received considerable attention. For instance,  $\mathcal{C}_8$  consists of cubic fourfolds containing a plane, whereas  $\mathcal{C}_{14}$  corresponds to cubic fourfolds containing a quintic del Pezzo surface, see [H2]. Relying on Fano's work [Fa], recently Bolognesi and Russo [BR] have shown that all fourfolds  $[X] \in \mathcal{C}_{14}$  are rational.

For every  $[X] \in \mathcal{C}$ , we denote by  $F(X) := \{\ell \in \mathbf{G}(1, 5) : \ell \subseteq X\}$  the Hilbert scheme of the lines contained in  $X$ . It is well known [BD] that  $F(X)$  is a hyperkähler fourfold deformation equivalent to the Hilbert square of a  $K3$  surface. For discriminant  $d = 2(n^2 + n + 1)$ , where  $n \geq 2$ , it is shown in [H1] that  $F(X)$  is *isomorphic* to the Hilbert scheme  $S^{[2]}$  of a polarized  $K3$  surface  $(S, H)$  with  $H^2 = d$ . If  $\mathcal{F}_g$  denotes the moduli space of polarized  $K3$  surfaces of genus  $g$ , the previous assignment induces a rational map

$$\mathcal{F}_{\frac{d}{2}+1} \dashrightarrow \mathcal{C}_d,$$

which is a birational isomorphism for  $d \equiv 2 \pmod{6}$  and a degree 2 cover for  $d \equiv 0 \pmod{6}$ . This map, though non-explicit for it is defined at the level of moduli spaces of weight-2 Hodge structures, opens the way to the study of  $\mathcal{F}_{n^2+n+2}$  via the concrete geometry of cubic fourfolds, without making a direct reference to  $K3$  surfaces! The main result of this paper concerns the universal  $K3$  surface  $\mathcal{F}_{g,1} \rightarrow \mathcal{F}_g$ .

**Theorem 1.1.** *The universal  $K3$  surface  $\mathcal{F}_{14,1}$  of genus 14 is rational.*

Nuer [Nu] proved that  $\mathcal{C}_{26}$  (and hence  $\mathcal{F}_{14}$  as well) is unirational. His proof relies on the fact that a general fourfold  $[X] \in \mathcal{C}_{26}$  contains certain smooth rational surfaces, whose

parameter space forms a unirational family. One can also show that  $\mathcal{C}_{44}$  is unirational, for a general  $[X] \in \mathcal{C}_{44}$  contains a Fano embedded Enriques surface and their moduli space is unirational, see [Ve2] and also [Nu]. Recently, Lai [L] showed that  $\mathcal{C}_{42}$  is uniruled.

Mukai in a celebrated series of papers [M1], [M2], [M3], [M4], [M5] established structure theorems for polarized  $K3$  surfaces of genus  $g \leq 12$ , as well as  $g = 13, 16, 18, 20$ . In particular,  $\mathcal{F}_g$  is unirational for those value of  $g$ . No structure theorem for the general  $K3$  surface of genus 14 is known. A quick inspection of Mukai's methods shows that the universal  $K3$  surface  $\mathcal{F}_{g,1}$  is unirational for  $g \leq 11$  as well. On the other hand, Gritsenko, Hulek and Sankaran [GHS] have proved that  $\mathcal{F}_g$  is a variety of general type for  $g > 62$ , as well as for  $g = 47, 51, 53, 55, 58, 59, 61$ . In a similar vein, recently it has been established in [TVA] that  $\mathcal{C}_d$  is of general type for all  $d$  sufficiently large. As pointed out in Remark 5.4, whenever  $\mathcal{F}_g$  is of general type, the Kodaira dimension of  $\mathcal{F}_{g,1}$  is equal to 19.

The proof of Theorem 1.1 relies on the connection between singular scrolls and special cubic fourfolds. We fix a general point  $[X] \in \mathcal{C}_{26}$  and denote by  $S$  the *associated*  $K3$  surface, such that  $S^{[2]} \cong F(X) \hookrightarrow \mathbf{G}(1, 5)$ . For each  $p \in S$ , we introduce the rational curve

$$\Delta_p := \{\xi \in S^{[2]} : \{p\} = \text{supp}(\xi)\}.$$

Under the Plücker embedding  $\mathbf{G}(1, 5) \subseteq \mathbf{P}^{14}$ , the degree of  $\Delta_p \subseteq F(X)$  is equal to 7, which suggests that each point of  $p \in S$  parametrizes a *septic* scroll  $R = R_p \subseteq X$ . Imposing the condition  $\text{disc}\langle h^2, [R] \rangle = 26$ , one obtains  $R^2 = 25$ . Assuming  $R$  has isolated non-normal nodal singularities, the double point formula implies that  $R$  has precisely 3 non-normal nodes. We shall prove that indeed, a general fourfold  $[X] \in \mathcal{C}_{26}$  carries a 2-dimensional family of 3-nodal scrolls  $R \subseteq X$  with  $\deg(R) = 7$ . Furthermore, this family of scrolls is parametrized by the  $K3$  surface  $S$  associated to  $X$ .

We now describe the moduli space of 3-nodal septic scrolls. We start with the Hirzebruch surface  $\mathbf{F}_1 := \text{Bl}_o(\mathbf{P}^2)$ , where  $o \in \mathbf{P}^2$ , and denote by  $\ell$  the class of a line and by  $E$  the exceptional divisor. The smooth septic scroll  $R' = S_{3,4} \subseteq \mathbf{P}^8$  is the image of the linear system

$$\phi_{|4\ell-3E|} : \mathbf{F}_1 \hookrightarrow \mathbf{P}^8.$$

We shall show in Section 3 that the secant variety  $\text{Sec}(R') \subseteq \mathbf{P}^8$  is as expected 5-dimensional. Choose general points  $a_1, a_2, a_3 \in \text{Sec}(R')$  and denote by  $\Lambda := \langle a_1, a_2, a_3 \rangle \in \mathbf{G}(2, 8)$  their linear span. The image  $R \subseteq \mathbf{P}^5$  of the projection with center  $\Lambda$

$$\pi_\Lambda : R' \rightarrow \mathbf{P}^5$$

is a 3-nodal septic scroll. Conversely, up to the action of  $PGL(6)$  on the ambient projective space  $\mathbf{P}^5$ , each such scroll appears in this way. We denote by  $\mathfrak{H}_{\text{scr}}$  the moduli space of unparametrized 3-nodal septic scrolls in  $\mathbf{P}^5$ , that is, the quotient of the corresponding Hilbert scheme under the action of  $PGL(6)$ . Then as showed in Proposition 3.6, the space  $\mathfrak{H}_{\text{scr}}$  turns out to be birationally isomorphic to the 9-dimensional unirational variety

$$\mathfrak{H}_{\text{scr}} \cong \text{Sym}^3(\text{Sec}(R'))/\text{Aut}(R').$$

Fix a general 3-nodal septic scroll  $R \subseteq \mathbf{P}^5$ . A general  $X \in \mathbf{P}(H^0(\mathcal{I}_{R/\mathbf{P}^5}(3))) = \mathbf{P}^{12}$  is a smooth cubic fourfold. Since  $R$  has no further singularities apart from the three non-normal nodes, the double point formula implies that  $[X] \in \mathcal{C}_{26}$ . One sets up the following incidence

correspondence between scrolls and cubic fourfolds of discriminant 26:

$$\begin{array}{ccc} \mathfrak{X} := \left\{ (X, R) : R \subseteq X \right\} / PGL(6) & & \\ \swarrow \pi_1 & & \searrow \pi_2 \\ \mathcal{C}_{26} & & \mathfrak{H}_{\text{scr}} \end{array}$$

Thus  $\mathfrak{X}$  is birational to a  $\mathbf{P}^{12}$ -bundle over the unirational variety  $\mathfrak{H}_{\text{scr}}$ . We then show that the fibre over a general cubic fourfold  $[X] \in \mathcal{C}_{26}$  of the projection  $\pi_1$  is 2-dimensional and isomorphic to the K3 surface  $S$  appearing in the identification  $F(X) \cong S^{[2]}$ . We summarize the discussion above.

**Theorem 1.2.** *The universal K3 surface  $\mathcal{F}_{14,1}$  is birational to the  $\mathbf{P}^{12}$ -bundle  $\mathfrak{X}$  over the moduli space  $\mathfrak{H}_{\text{scr}}$  of 3-nodal septic scrolls  $R \subseteq \mathbf{P}^5$ . A general fourfold  $[X] \in \mathcal{C}_{26}$  contains a two-dimensional family of such scrolls  $R \subseteq X \subseteq \mathbf{P}^5$ . The space of such scrolls is isomorphic to the K3 surface associated to  $X$ .*

Theorem 1.2 allows us to elucidate the structure of  $\mathcal{F}_{14,1}$  even further and prove its rationality. We fix a 3-nodal septic scroll  $R \subseteq \mathbf{P}^5$  as above and denote its nodes by  $p_1, p_2, p_3$ . The curve  $\Gamma_R \subseteq \mathbf{G}(1, 5)$  induced by the rulings of  $R$  is a smooth rational septic curve admitting bisecant lines  $L_1, L_2$  and  $L_3$  in the Plücker embedding of  $\mathbf{G}(1, 5)$ . Precisely,  $L_i$  parametrizes the lines passing through  $p_i$  and contained in the 2-plane  $P_i$  spanned by the two rulings of  $R$  that intersect at the node  $p_i$ , for  $i = 1, 2, 3$ . Since  $\Gamma_R$  spans a 7-dimensional linear space in projective space  $\mathbf{P}^{14}$  containing  $\mathbf{G}(1, 5)$ , using Mukai's work [M6] on realizing canonical genus 8 curves as linear sections of the Grassmannian  $\mathbf{G}(1, 5)$ , it follows that the intersection  $\mathbf{G}(1, 5) \cdot \langle \Gamma_R \rangle$  is a semi-stable curve of genus 8. We denote by  $Q \subseteq \langle \Gamma_R \rangle = \mathbf{P}^7$  the residual curve defined by the following equality:

$$(1) \quad \mathbf{G}(1, 5) \cdot \langle \Gamma_R \rangle = \Gamma_R + L_1 + L_2 + L_3 + Q.$$

We shall establish in Lemmas 4.1 and 4.2 that  $Q$  is a smooth rational quartic curve and  $Q \cdot L_i = 1$  for  $i = 1, 2, 3$ , as well as  $Q \cdot \Gamma_R = 3$ . Therefore  $Q$  is the curve of rulings of a quartic scroll  $R_Q \subseteq \mathbf{P}^5$ , which contains three rulings  $\ell_1, \ell_2, \ell_3$ , such that that  $p_i \in \ell_i$  and  $\ell_i \in P_i$  for  $i = 1, 2, 3$ . In particular,  $R_Q$  contains the three nodes of the septic scroll  $R$ . We can show furthermore that  $R_Q$  is smooth and isomorphic to  $F_0$ , see Theorem 4.10.

The construction above can be reversed. Using the automorphism group of the scroll  $R_Q \subseteq \mathbf{P}^5$ , we fix three of its rulings  $\ell_1, \ell_2, \ell_3 \in \mathbf{G}(1, 5)$ , as well as points  $p_i \in \ell_i$ . We set

$$\mathbf{P}_i^3 := \{P_i \in \mathbf{G}(2, 5) : \ell_i \subseteq P_i\},$$

for  $i = 1, 2, 3$ , then define a map

$$\varkappa : \mathbf{P}_1^3 \times \mathbf{P}_2^3 \times \mathbf{P}_3^3 / \mathfrak{S}_3 \dashrightarrow \mathfrak{H}_{\text{scr}},$$

by reversing the above construction and using the decomposition (1). Along with the fixed point  $p_i$ , each 2-plane  $P_i \in \mathbf{P}_i^3$  defines a line  $L_i \subseteq \mathbf{G}(1, 5)$  meeting the curve  $Q$  at the point  $\ell_i$ . Precisely,  $L_i$  is the line of lines in  $P_i$  passing through the point  $p_i$ . To the triple  $(P_1, P_2, P_3)$  we associate the scroll  $R \subseteq \mathbf{P}^5$  whose associated curve of rulings  $\Gamma_R$  is defined by the formula (1). The above discussion indicates that  $\varkappa$  is dominant. In fact more can be proved:

**Theorem 1.3.** *The moduli space of scrolls  $\mathfrak{H}_{\text{scr}}$  is birational to  $\mathbf{P}_1^3 \times \mathbf{P}_2^3 \times \mathbf{P}_3^3 / \mathfrak{S}_3$  and is thus rational.*

Indeed, using the theorem on symmetric functions, see [Ma] or [GKZ] Theorem 2.8 for a recent reference, all symmetric products of projective spaces are known to be rational. It is now clear that Theorem 1.3 coupled with Theorem 1.2 implies that  $\mathcal{F}_{14,1}$  is a rational variety.

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## 2. $K3$ SURFACES AND CUBIC FOURFOLDS

We begin by setting some notation. Let  $U \subseteq |\mathcal{O}_{\mathbb{P}^5}(3)|$  be the locus of smooth cubic fourfolds and set

$$\mathcal{C} := U/PGL(6)$$

to be the 20-dimensional moduli space of cubic fourfolds. For an integer  $d \equiv 0, 2 \pmod{6}$ , as pointed out in the Introduction,  $\mathcal{C}_d$  denotes the irreducible divisor of  $\mathcal{C}$  consisting of special cubic fourfolds of discriminant  $d$ . As usual,  $\mathcal{F}_g$  is the irreducible 19-dimensional moduli space of smooth polarized  $K3$  surfaces  $(S, H)$  of genus  $g$ , that is, with  $H^2 = 2g - 2$ . We denote by  $u : \mathcal{F}_{g,1} \rightarrow \mathcal{F}_g$  the universal  $K3$  surface of genus  $g$  in the sense of stacks. Each fibre  $u^{-1}([S, H])$  is identified with the  $K3$  surface  $S$ .

Using the Hodge-theoretic similarity between  $K3$  surfaces of genus  $g = n^2 + n + 1$  and special cubic fourfolds of degree  $2g - 2$ , Hassett [H1] constructed a morphism of moduli spaces

$$\varphi : \mathcal{F}_{n^2+n+2} \rightarrow \mathcal{C}_{2(n^2+n+1)},$$

which is birational for  $n \equiv 0, 2 \pmod{3}$ , and of degree 2 for  $n \equiv 1 \pmod{3}$  respectively. In particular, for  $n = 3$  there is a birational isomorphism of spaces of weight 2 Hodge structures

$$\varphi : \mathcal{F}_{14} \xrightarrow{\cong} \mathcal{C}_{26},$$

that will be of use throughout the paper. At the moment, there is no geometric construction of the polarized  $K3$  surface  $\varphi^{-1}([X])$  associated to a general fourfold  $[X] \in \mathcal{C}_{26}$ .

We recall basic facts on Hilbert squares of  $K3$  surfaces and refer to [HT1] for a general reference on these matters. Let  $(S, H)$  be a polarized  $K3$  surface with  $\text{Pic}(S) = \mathbb{Z} \cdot H$  and  $H^2 = 2g - 2$ . We denote by  $S^{[2]}$  the Hilbert scheme of length two 0-dimensional subschemes on  $S$ . Then  $H^2(S^{[2]}, \mathbb{Z})$  is endowed with the *Beauville-Bogomolov* quadratic form  $q$ . We denote by  $\Delta \subseteq S^{[2]}$  the diagonal divisor consisting of zero-dimensional subschemes supported only at a single point and by  $\delta := \frac{[\Delta]}{2} \in H^2(S^{[2]}, \mathbb{Z})$  the reduced diagonal class. One has  $q(\delta, \delta) = -2$ . Note the canonical identification

$$\Delta = \mathbf{P}(T_S) = \cup \{\Delta_p : p \in S\},$$

where  $\Delta_p$  is the rational curve consisting of those 0-dimensional subschemes  $\xi \in \Delta$  such that  $\text{supp}(\xi) = \{p\}$ . We set  $\delta_p := [\Delta_p] \in H_2(S^{[2]}, \mathbb{Z})$ .

For a curve  $C \in |H|$  in the polarization class, we introduce the divisor

$$f_C := \{\xi \in S^{[2]} : \text{supp}(\xi) \cap C \neq \emptyset\}$$

and set  $f := [f_C] \in H^2(S^{[2]}, \mathbb{Z})$ . If  $p \in S$  is a general point, we also define the curve

$$F_p := \{\xi = p + x \in S^{[2]} : x \in C\}$$

and set  $f_p := [F_p] \in H_2(S^{[2]}, \mathbb{Z})$ . The Beauville-Bogomolov form can be extended to a quadratic form on  $H_2(S^{[2]}, \mathbb{Z})$ , by setting  $q(\alpha, \alpha) := q(w_\alpha, w_\alpha)$ , with  $w_\alpha \in H^2(S^{[2]}, \mathbb{Z})$  being the

class characterized by the property  $\alpha \cdot u = q(w_\alpha, u)$ , for every  $u \in H^2(S^{[2]}, \mathbb{Z})$ . Here  $\alpha \cdot u$  denotes the usual intersection product.

One has the following decompositions, orthogonal with respect to  $q$ , both for the Picard group and for the group  $N_1(S^{[2]}, \mathbb{Z})$  of 1-cycles modulo numerical equivalence:

$$\text{Pic}(S^{[2]}) \cong \mathbb{Z} \cdot f \oplus \mathbb{Z} \cdot \delta \quad \text{and} \quad N_1(S^{[2]}, \mathbb{Z}) \cong \mathbb{Z} \cdot f_p \oplus \mathbb{Z} \cdot \delta_p.$$

We record, the more or less obvious relations:

$$(2) \quad f \cdot f_p = 2g - 2, \quad \delta \cdot \delta_p = -1, \quad f \cdot \delta_p = 0 \quad \text{and} \quad \delta \cdot f_p = 0.$$

Assume now that  $X \subseteq \mathbf{P}^5$  is a general special cubic fourfold of discriminant 26 and let  $[S, H] = \varphi^{-1}([X]) \in \mathcal{F}_{14}$  be the associated polarized K3 surface such that

$$(3) \quad S^{[2]} \cong F(X) \subseteq \mathbf{G}(1, 5) \hookrightarrow \mathbf{P}^{14}.$$

Following [BD], let  $\gamma_S := [\mathcal{O}_{S^{[2]}}(1)]$  be the hyperplane class of  $\mathbf{G}(1, 5)$  restricted to the Hilbert square under the identification (3). Since  $q(\gamma_S, \gamma_S) = 6$ , using (2), it quickly follows that

$$\gamma_S = 2f - 7\delta \in H^2(S^{[2]}, \mathbb{Z}).$$

**Proposition 2.1.** *Suppose  $[S, H] \in \mathcal{F}_{26}$  is a general element and let  $R \subseteq S^{[2]}$  be an effective 1-cycle such that  $R \cdot \gamma_S = 7$ . Then  $R$  is one of the rational irreducible curves  $\Delta_p$ , for  $p \in S$ . In particular,  $R$  is smooth.*

*Proof.* Assume that  $R$  is an effective 1-cycle and write  $[R] = af_p - b\delta_p \in N_1(S^{[2]}, \mathbb{Z})$ . Since  $7 = R \cdot \gamma_S = 52a - 7b$ , hence we can write  $a = 7a_1$ , with  $a_1 \in \mathbb{Z}$ , and then  $b = 52a_1 - 1$ . Using [BM] Proposition 12.6, we have  $q(R, R) \geq -\frac{5}{2}$ . We obtain  $39a_1^2 - 26a_1 - 1 \leq 0$ , and the only integer solution of this inequality is  $a_1 = 0$ , therefore  $[R] = \delta_p$ .

Since  $[R] \cdot \delta = -1$ , it follows that  $R \subseteq \Delta$ . We claim that  $R$  lies in one of the fibres of the  $\mathbf{P}^1$ -bundle  $\pi : \Delta = \mathbf{P}(T_S) \rightarrow S$ , which implies that  $R = \Delta_p$ , for some  $p \in S$ . Indeed, otherwise  $\pi(R) \equiv mH$ , for some  $m > 0$ . Accordingly, we write

$$mH^2 = R \cdot \pi^{-1}(H) = R \cdot f = \delta_p \cdot f = 0,$$

which is a contradiction.  $\square$

**Remark 2.2.** Unlike degree 26, for other values of  $d$ , a general  $[X] \in \mathcal{C}_d$  may contain several types of scrolls. For instance when  $d = 14$  and  $\gamma_S = 2f - 5\delta$ , the curves  $\Delta_p$  with  $p \in S$  correspond to quintic scroll, but  $X$  also contains quartic scrolls corresponding to rational curves  $R \subseteq F(X)$  with  $[R] = 3f_p - 16\delta_p$ . Note that  $q(R, R) = -2$ .

We now recall the correspondence between scrolls and rational curves in Grassmannians. Following for instance [Dol] 10.4, we define a *rational scroll* to be the image  $R \subseteq \mathbf{P}^n$  of a  $\mathbf{P}^1$ -bundle  $\pi : R' = \mathbf{P}(\mathcal{E}) \rightarrow \mathbf{P}^1$  under a map  $\phi : R' \rightarrow \mathbf{P}^n$  given by a linear subsystem of  $|\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)|$ , thus sending the fibres of  $\pi$  to lines in  $\mathbf{P}^n$ . Let  $f_R : \mathbf{P}^1 \rightarrow \mathbf{G}(1, n)$  be the map

$$f_R(t) := [\phi(\pi^{-1}(t))]$$

and denote by  $\Gamma_R$  its image. Conversely, start with a non-degenerate map  $f : \mathbf{P}^1 \rightarrow \mathbf{G}(1, n)$ , then consider the pull-back under  $f$  of the projectivization of tautological rank 2 vector over  $\mathbf{G}(1, n)$ , that is,

$$(4) \quad \Xi := \left\{ (t, x) : t \in \mathbf{P}^1, x \in L_{f(t)} \right\} \subseteq \mathbf{P}^1 \times \mathbf{P}^n.$$

Here  $L_{f(t)} \subseteq \mathbf{P}^n$  denotes the line whose moduli point in  $\mathbf{G}(1, n)$  is precisely  $f(t)$ .

The projection  $\pi_2 : \Xi \rightarrow \mathbf{P}^n$  is a finite map and its image is a scroll  $R \subseteq \mathbf{P}^n$  of degree

$$\deg(\Gamma_R) = \deg f^* \left( \mathcal{O}_{\mathbf{G}(1, n)}(1) \right).$$

Throughout the paper, we interpret scrolls in terms of their associated curves of rulings. It will be useful to determine, using this language, when a scroll is smooth.

**Proposition 2.3.** *Let  $R \subseteq \mathbf{P}^n$  be a scroll which is not a cone and such that  $\Gamma_R$  is a smooth rational curve in  $\mathbf{G}(1, n)$  which is not contained in a plane. Then there is a bijective correspondence between singularities of  $R$  and bisecant lines to  $\Gamma_R$  lying on  $\mathbf{G}(1, n)$ . In particular, if  $\Gamma_R$  admits no bisecant lines contained in  $\mathbf{G}(1, n)$ , then  $R$  is smooth.*

*Proof.* We consider the projection  $\pi_2 : \Xi \rightarrow R$  defined by (4). Then  $\Xi$  is a smooth variety and the assumptions made on  $R$  imply that  $\pi_2$  is a finite map. If a point  $x \in R$  corresponds to a singularity, then one of the two following possibilities occur: (i) the fibre  $\pi_2^{-1}(x)$  consists of more than point, or (ii) the differential of  $\pi_2$  at a point of  $(t, x) \in \pi_2^{-1}(x)$  is not an isomorphism.

In case (i), we choose distinct points  $t_1, t_2 \in \pi_1(\pi_2^{-1}(x))$ . Denoting by  $\ell_1 := f_R(t_1)$  and  $\ell_2 := f_R(t_2)$  the rulings of  $\Xi$  corresponding to these points, we observe that  $x \in \ell_1 \cap \ell_2$ . The set  $L$  of lines in the 2-plane  $\langle \ell_1, \ell_2 \rangle$  passing through  $x$  is a line in  $\mathbf{G}(1, n)$  such that  $\Gamma_R \cap L \supseteq \{\ell_1, \ell_2\}$ , that is,  $\Gamma_R$  possesses a secant line lying inside  $\mathbf{G}(1, n)$  in its Plücker embedding. Note that  $L$  is a genuine secant line in the sense that it meets the curve  $\Gamma_R$  in two distinct points  $\ell_1$  and  $\ell_2$ . All lines lying inside  $\mathbf{G}(1, n)$  in its Plücker embedding correspond to pencils of lines in a 2-plane passing through a point in  $\mathbf{P}^n$ . Thus conversely, when such a line meets  $\Gamma_R$  in two distinct points, these will correspond to two incident rulings of  $R$ . In particular  $R$  is singular at their point of intersection.

To deal with case (ii), we carry out a local calculation. Assume  $(t_0, x) \in \Xi$  is a point at which the differential of  $\pi_2$  is not an isomorphism. We set  $\ell_0 := f_R(t_0)$  and denote by

$$p_{ij}(t) = a_i(t)b_j(t) - a_j(t)b_i(t), \quad \text{where } 0 \leq i < j \leq n$$

the Plücker coordinates of the curve  $\Gamma_R$  in a neighborhood of  $\ell_0$ , where  $a(t) = (a_0(t), \dots, a_n(t))$  and  $b(t) = (b_0(t), \dots, b_n(t))$ .

In local coordinates, the map  $\pi_2$  is given by  $\mathbf{P}^1 \times \mathbb{C} \ni ([\lambda, \mu], t) \mapsto [(\lambda a_i(t) + \mu b_i(t))] =: x$ . By direct calculation, the condition that  $(d\pi_2)_{(t_0, x)}$  is not an isomorphism is equivalent to

$$b'(t_0) \wedge a(t_0) = 0 \in \bigwedge^2 \mathbb{C}^{n+1}.$$

Setting  $a_i := a_i(t_0)$ ,  $b_i := b_i(t_0)$ ,  $a'_i := a'_i(t_0)$  and  $b'_i := b'_i(t_0)$ , we then observe that the Plücker coordinates of a point on the tangent line  $\mathbb{T}_{\ell_0}(\Gamma_R) \subseteq \mathbf{P}^{\binom{n+1}{2}-1}$  are given by

$$a_i b_j - a_j b_i + \mu(a'_i b_j + a_i b'_j - a'_j b_i - a_j b'_i) = b_j(a_i + \mu a'_i) - b_i(a_j + \mu a'_j),$$

for some scalar  $\mu$ . It follows that the tangent line to  $\Gamma_R$  at  $\ell_0$  is contained in  $\mathbf{G}(1, n)$ . The argument being reversible, we finish the proof.  $\square$

The scrolls  $R \subseteq \mathbf{P}^n$  we consider most of the time have at worst *non-normal nodal singularities*  $x \in R$ , corresponding to the case  $|\phi^{-1}(x)| = 2$ . The tangent cone of  $R$  at  $x$  is isomorphic to the union of two 2-planes in  $\mathbf{P}^4$  meeting in one point. According to Proposition 2.3, to each

such singularity corresponds a line in the Plücker embedding of  $\mathbf{G}(1, n)$  meeting  $\Gamma_R$  in two distinct points.

Suppose now that  $R \subseteq X \subseteq \mathbf{P}^5$  is a rational scroll with isolated nodal singularities contained in a cubic fourfold. Using the *double point formula* [Ful] 9.3 applied to the map  $\phi : R' \rightarrow X$ , we find the number of singularities of  $R = \phi(R')$ :

$$(5) \quad D(\phi) = R^2 - 6h^2 - K_R^2 - 3h \cdot K_R + 2\chi_{\text{top}}(R).$$

When  $[X] \in \mathcal{C}_{26}$ , assuming that  $A(X) = \langle h^2, [R] \rangle$ , where  $h^2 \cdot [R] = \deg(R) = 7$ , necessarily  $R^2 = 25$ . From formula (5), we compute  $D(\phi) = 3$ , that is, if  $R$  has only (isolated) improper nodes, then it is 3-nodal.

Before stating our next result, we recall that  $\overline{\mathcal{M}}_0(F(X), 7)$  denotes the space of stable maps  $f : C \rightarrow F(X)$ , from a nodal curve  $C$  of genus zero such that  $\deg(f^*(\mathcal{O}_{F(X)}(1))) = 7$ . We denote by  $\mathcal{M}_0(F(X), 7)$  the open sublocus consisting of maps with source  $\mathbf{P}^1$  and denote by  $\overline{\mathcal{M}}_7(X)$  the closure of  $\mathcal{M}_0(F(X), 7)$  inside  $\overline{\mathcal{M}}_0(F(X), 7)$ .

**Corollary 2.4.** *Let  $[X] \in \mathcal{C}_{26}$  a general special fourfold of discriminant 26 and  $[S, H] \in \mathcal{F}_{26}$  its associated K3 surface. Then there is an isomorphism  $S \cong \overline{\mathcal{M}}_7(X)$ .*

*Proof.* Using the identification  $S^{[2]} \cong F(X)$ , we define the map  $j : S \rightarrow \overline{\mathcal{M}}_7(X)$ , by setting  $j(p) := \Delta_p \subseteq F(X)$ . All points in the image of  $j$  consist of embedded smooth rational curves  $\mathbf{P}^1 \xrightarrow{\cong} \Delta_p$  and we identify  $\Delta_p$  with the corresponding map  $\mathbf{P}^1 \hookrightarrow F(X)$ . In a neighborhood of this map, the moduli space  $\overline{\mathcal{M}}_0(F(X), 7)$  is locally isomorphic to the Hilbert scheme of septic rational curves on  $F(X)$ .

The tangent space of  $\overline{\mathcal{M}}_7(X)$  at the point  $[\Delta_p]$  is canonically isomorphic to  $H^0(N_{\Delta_p/F(X)})$ . Using the following exact sequence on  $\Delta_p \cong \mathbf{P}^1$

$$0 \longrightarrow N_{\Delta_p/\Delta} \longrightarrow N_{\Delta_p/F(X)} \longrightarrow \mathcal{O}_{\Delta_p}(\Delta) \longrightarrow 0,$$

since  $N_{\Delta_p/\Delta} = \mathcal{O}_{\Delta_p}^{\oplus 2}$  and  $\mathcal{O}_{\Delta_p}(\Delta) = \mathcal{O}_{\Delta_p}(-1)$ , we compute  $N_{\Delta_p/F(X)} = \mathcal{O}_{\Delta_p}^{\oplus 2} \oplus \mathcal{O}_{\Delta_p}(-1)$ . It follows that  $H^1(\Delta_p, N_{\Delta_p/F(X)}) = 0$ , hence the obstruction space for deformations vanishes and

$$\dim T_{[\Delta_p]}(\overline{\mathcal{M}}_0(F(X), 7) = h^0(\Delta_p, N_{\Delta_p/F(X)}) = 2.$$

We conclude that  $[\Delta_p]$  is a smooth point of expected dimension of  $\overline{\mathcal{M}}_7(X)$ , for every  $p \in S$ .

Furthermore,  $j$  is injective, because for distinct points  $p, q \in S$ , since  $\Delta_p \cap \Delta_q = \emptyset$ , the associated scrolls  $R_p$  and  $R_q$  share no rulings. We finally observe that  $j$  is an immersion. Indeed, for each  $p \in S$ , we have the identification  $\Delta_p = \mathbf{P}(T_p(S) \oplus T_p(S)/T_p(S))$ , the quotient being given by the diagonal embedding. Thus the differential  $dj(p)$  is essentially the identity map, via the identification  $\mathbf{P}(T_S) \cong \bigcup_{p \in S} \mathbf{P}(N_{\Delta_p/\Delta})$ . Since according to Proposition 2.1, we have that  $\mathcal{M}_0(F(X), 7) \subseteq \text{Im}(j)$ , we can conclude the proof.  $\square$

### 3. NODAL SEPTIC SCROLLS AND CUBIC FOURFOLDS

In this section we study in more detail the moduli space  $\mathfrak{H}_{\text{scr}}$  of 3-nodal septic scrolls that will be used to parametrize the universal K3 surface of degree 26. We fix once and for all the smooth septic scroll

$$R' := S_{3,4} \hookrightarrow \mathbf{P}^8,$$

given as the image of the map  $\phi_{|4\ell-3E|}$  on the Hirzebruch surface  $F_1 = \text{Bl}_o(\mathbf{P}^2)$ . We denote by  $h : R' \rightarrow \mathbf{P}^1$  the map induced by the linear system  $|\ell - E|$ . The fibres of  $h$  are pairwise disjoint lines in  $\mathbf{P}^8$ . Equivalently, we consider the vector bundle on  $\mathbf{P}^1$

$$\mathcal{G} = \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(4)$$

and then  $R' \cong \mathbf{P}(\mathcal{G})$ . One has the canonical identification between space of sections:

$$H^0(R', \mathcal{O}_{R'}(1)) \cong H^0(\mathbf{P}(\mathcal{G}), \mathcal{O}_{\mathbf{P}(\mathcal{G})}(1)) \cong H^0(\mathbf{P}^1, \mathcal{G}).$$

Later, when computing the dimension of the parameter space of 3-nodal septic scrolls, we shall make use of the basic fact

$$\dim \text{Aut}(R') = \dim \text{Aut}(F_1) = 6.$$

Every smooth septic scroll in  $\mathbf{P}^8$  is obtained from  $R'$  by applying a linear transformation of  $\mathbf{P}^8$ . In particular, the Hilbert scheme of septic scrolls in  $\mathbf{P}^8$  has dimension equal to

$$\dim PGL(9) - \dim \text{Aut}(R') = 80 - 6 = 74.$$

Using coordinates in  $\mathbf{P}^8$ , if  $\mathbf{P}_{y_0, \dots, y_3}^3 \subseteq \mathbf{P}^8$  is the linear span of the twisted cubic  $E$  corresponding to the exceptional divisor on  $F_1$  and  $\mathbf{P}_{x_0, \dots, x_4}^4 \subseteq \mathbf{P}^8$  is the linear span of a rational quartic curve linearly equivalent to  $\ell$ , then the ideal of  $R'$  in  $\mathbf{P}^8$  is given by the following determinantal condition, see for instance [Ha] Lecture 9:

$$\text{rk} \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & y_0 & y_1 & y_2 \\ x_1 & x_2 & x_3 & x_4 & y_1 & y_2 & y_3 \end{pmatrix} \leq 1.$$

The secant variety  $\text{Sec}(R') \subseteq \mathbf{P}^8$  is also determinantal, with equations given by the  $3 \times 3$  minors of the following 1-generic matrix:

$$\text{rk} \begin{pmatrix} x_0 & x_1 & x_2 & y_0 & y_1 \\ x_1 & x_2 & x_3 & y_1 & y_2 \\ x_2 & x_3 & x_4 & y_2 & y_3 \end{pmatrix} \leq 2$$

It follows from [CJ] Lemma 3.1 that, as expected,  $\text{Sec}(R')$  is 5-dimensional. Furthermore, applying e.g. [Ei] Corollary 3.3, it follows that the singular locus of  $\text{Sec}(R')$  coincides with the scroll  $R'$ .

**Lemma 3.1.** *Let  $a_1, a_2, a_3 \in \text{Sec}(R')$  be general points and set  $\Lambda := \langle a_1, a_2, a_3 \rangle \in \mathbf{G}(2, 8)$ . The image  $R$  of the projection  $\pi : R' \rightarrow \mathbf{P}^5$  with center  $\Lambda$  has three non-normal nodes corresponding to the three bisecant lines passing through  $a_1, a_2$  and  $a_3$  and no further singularities.*

*Proof.* The chosen points  $a_1, a_2, a_3$  can be assumed to lie in  $\text{Sec}(R') - (R' \cup \text{Tan}(R'))$ . Since  $\dim \text{Sec}(R') = 5$ , by using the *Trisecant lemma*, see for instance [CC] Proposition 2.6, it follows that the scheme-theoretic intersection of  $\text{Sec}(R')$  with  $\Lambda$  consists only of the points  $a_1, a_2, a_3$ . In particular,  $\Lambda \cap R' = \emptyset$ , hence the projection  $\pi = \pi_\Lambda : R' \rightarrow R$  is a regular morphism. Furthermore, each point  $a_i$  lies on a unique bisecant line  $\langle x_i, y_i \rangle$ , where  $x_i$  and  $y_i$  are distinct points of  $R'$ , for  $i = 1, 2, 3$ .

Suppose now that for  $x, y \in R'$ , one has  $\pi(x) = \pi(y)$ . This happens if and only if  $\langle x, y \rangle \cap \Lambda \neq \emptyset$ , hence  $\emptyset \neq \langle x, y \rangle \cap \Lambda \subseteq \{a_1, a_2, a_3\}$  and then necessarily  $\{x, y\} = \{x_i, y_i\}$ , for  $i \in \{1, 2, 3\}$ . Since  $\Lambda \cap \text{Tan}(R') = \emptyset$ , it follows that the differential of  $\pi$  is everywhere injective. To summarize, the only singularities of  $R$  are the three non-normal nodes  $\pi(x_i) = \pi(y_i)$ , for  $i = 1, 2, 3$ .  $\square$



We now fix a general projection  $\pi = \pi_\Lambda : R' \rightarrow \mathbf{P}^5$  as in Lemma 3.1. We denote by  $p_i$  the three singularities of the image scroll  $R$ . The map  $\pi_\Lambda$  is defined by the 6-dimensional subspace  $V := H^0(\mathbf{P}^8, \mathcal{I}_{\Lambda/\mathbf{P}^8}(1))$  of  $H^0(\mathbf{P}^1, \mathcal{G})$ . To give  $\Lambda$  amounts to specifying  $V \subseteq H^0(\mathbf{P}^1, \mathcal{G})$ . Since  $\Lambda \cap R' = \emptyset$ , it follows that the evaluation map  $\text{ev}_V : V \otimes \mathcal{O}_{\mathbf{P}^1} \rightarrow \mathcal{G}$  is surjective. Hence  $\text{ev}_V$  defines a morphism

$$f : \mathbf{P}^1 \rightarrow \mathbf{G}(1, 5).$$

This map is induced by the ruling of the image scroll  $R$ , that is,  $f_R = f$  is the map given by  $f_R(t) := [\pi(h^{-1}(t))]$ , for  $t \in \mathbf{P}^1$ . Set  $\Gamma_R := \text{Im}(f_R)$ .

**Proposition 3.2.** *For a general choice of the 3-secant plane  $\Lambda$  to  $\text{Sec}(R')$ , the following hold:*

- (i)  $\dim \langle p_1, p_2, p_3 \rangle = 2$ .
- (ii)  $\langle p_1, p_2, p_3 \rangle \cap R = \{p_1, p_2, p_3\}$ .

*Proof.* It suffices to consider a codimension 2 general linear section  $Z \subseteq R' \subseteq \mathbf{P}^8$ . Then  $Z$  is a smooth 0-dimensional scheme supported at seven distinct points  $x_1, y_1, x_2, y_2, x_3, y_3$  and  $z$ , spanning a 6-dimensional linear space in  $\mathbf{P}^8$ . In particular,  $z$  does not lie in the 5-plane spanned by the points  $\{x_i, y_i\}_{i=1}^3$  and no line intersecting the lines  $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle, \langle x_3, y_3 \rangle$  exists. Pick general points  $a_i \in \langle x_i, y_i \rangle$ , for  $i = 1, 2, 3$ . Then the projection  $\pi_\Lambda$  defined by the plane  $\Lambda = \langle a_1, a_2, a_3 \rangle$  satisfies both conditions (i) and (ii).  $\square$

For a projection  $\pi_\Lambda$  satisfying the assumptions of Lemma 3.1, the map  $f_R : \mathbf{P}^1 \rightarrow \mathbf{G}(1, 5)$  is an embedding, for  $\Lambda$  intersects no ruling of  $R'$ . We record the conclusion of Proposition 2.3 for a scroll  $R$  as above:

**Proposition 3.3.** *The rational curve  $\Gamma_R \subseteq \mathbf{G}(1, 5)$  admits three secant lines that lie in  $\mathbf{G}(1, 5)$ . Conversely, a rational septic curve  $\Gamma \subseteq \mathbf{G}(1, 5)$  having three secant lines lying in  $\mathbf{G}(1, 5)$  is the curve of rulings of a 3-nodal septic scroll in  $\mathbf{P}^5$ .*

We establish a couple of properties concerning the linear system of cubic fourfolds containing a 3-nodal septic scroll:

**Proposition 3.4.** *The following statements hold for a general 3-nodal septic scroll  $R \subset \mathbf{P}^5$ :*

$$(i) \dim |\mathcal{I}_{R/\mathbf{P}^5}(3)| = 12 \quad \text{and} \quad (ii) \quad H^1(\mathbf{P}^5, \mathcal{I}_{R/\mathbf{P}^5}(3)) = 0.$$

*Proof.* Recall that  $R$  is the image of a projection  $\pi = \pi_\Lambda : R' \rightarrow R$  with center  $\Lambda$ , and denote by  $p_1, p_2, p_3 \in R$  the three (non-normal) singularities of  $R$  and by  $\{x_i, y_i\} = \pi^{-1}(p_i)$ , for  $i = 1, 2, 3$ . By Proposition 3.2, the points  $p_1, p_2$  and  $p_3$  are in general linear position in  $\mathbf{P}^5$  and thus impose independent conditions on cubic hypersurfaces, that is,  $H^1(\mathbf{P}^5, \mathcal{I}_{\text{Sing}(R)/\mathbf{P}^5}(3)) = 0$ .

By passing to cohomology in the short exact sequence

$$0 \longrightarrow \mathcal{I}_{R/\mathbf{P}^5}(3) \longrightarrow \mathcal{I}_{\text{Sing}(R)/\mathbf{P}^5}(3) \longrightarrow \mathcal{I}_{\text{Sing}(R)/R}(3) \longrightarrow 0,$$

we write the following exact sequence:

$$0 \longrightarrow H^0(\mathcal{I}_{R/\mathbf{P}^5}(3)) \longrightarrow H^0(\mathcal{I}_{\text{Sing}(R)/\mathbf{P}^5}(3)) \longrightarrow H^0(\mathcal{I}_{\text{Sing}(R)/R}(3)) \longrightarrow H^1(\mathcal{I}_{R/\mathbf{P}^5}(3)) \longrightarrow 0.$$

Clearly  $h^0(\mathbf{P}^5, \mathcal{I}_{\text{Sing}(R)/\mathbf{P}^5}(3)) = \binom{8}{3} - 3 = 53$ . Furthermore, we have the following identification of linear systems:

$$(6) \quad \pi^* \left( |\mathcal{I}_{\text{Sing}(R)/R}(3)| \right) = \left| \mathcal{I}_{\{x_1, y_1, x_2, y_2, x_3, y_3\}/R'}(12\ell - 9E) \right|.$$

The scroll  $[R] \in \mathfrak{H}_{\text{scr}}$  is obtained as a general projection like in Lemma 3.1. In particular, the points  $\{x_i, y_i\}_{i=1}^3 \subseteq R'$  are general as well, hence impose independent conditions on the linear system  $|12\ell - 9E|$  on  $R'$ . Using the identification (6), we compute:

$$h^0(R, \mathcal{I}_{\text{Sing}(R)/R}(3)) = h^0(R', \mathcal{O}_{R'}(12\ell - 9E)) - 6 = h^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(12)) - \binom{10}{2} - 6 = 40.$$

Therefore  $h^0(\mathbf{P}^5, \mathcal{I}_{R/\mathbf{P}^5}(3)) = 13$  if and only if  $h^1(\mathbf{P}^5, \mathcal{I}_{R/\mathbf{P}^5}(3)) = 0$ . This last statement can be proved via a simple *Macaulay* calculation by choosing the points  $a_1, a_2, a_3$  randomly in the variety  $\text{Sec}(R')$  whose equations have been given explicitly.  $\square$

**Remark 3.5.** It is possible to realize the rational curve  $\Gamma_R$  inside the linear system  $|\mathcal{O}_R(1)|$  as follows. Recall that we have denoted by  $\phi : \mathbf{F}_1 \hookrightarrow \mathbf{P}^8$  the embedding whose image is the smooth scroll  $R'$ . In  $|4\ell - 3E| \cong \mathbf{P}^8$ , we consider the space of reducible hyperplane sections:

$$\left\{ A' + L' : A' \in |3\ell - 2E|, L' \in |\ell - E| \right\}.$$

Note that  $L'$  is a ruling of  $R'$ , whereas  $A' \subseteq \mathbf{P}^8$  is a sextic with  $\langle A' \rangle = \mathbf{P}^6$  and with  $L' \cdot A' = 1$ . In the linear system  $|3\ell - 2E|$  there exists a *unique* sextic  $A'_0$  such that  $\Lambda \subseteq \langle A'_0 \rangle \subseteq \mathbf{P}^8$ . The curve  $A'_0$  corresponds to the unique curve in the linear system

$$\left| \mathcal{I}_{\{x_1, y_1, x_2, y_2, x_3, y_3\}/R'}(3\ell - 2E) \right|$$

on  $R'$ . Indeed,  $x_i, y_i \in A'_0$ , therefore  $a_i \in \langle x_i, y_i \rangle \subseteq \langle A'_0 \rangle$ , for  $i = 1, 2, 3$ . It then follows that  $\Lambda = \langle a_1, a_2, a_3 \rangle \subseteq \langle A'_0 \rangle$ . The projection  $A_0 := \pi(A'_0) \subseteq \mathbf{P}^5$  is a sextic curve on  $R$  passing through the nodes  $p_1, p_2, p_3$ . One identifies  $\Gamma_R$  with  $A_0$  via the map  $L \mapsto L \cdot A_0$ .

We denote by  $\mathcal{H}_{\text{scr}}$  the Hilbert scheme of 3-nodal septic scrolls in  $R \subseteq \mathbf{P}^5$  and set

$$\mathfrak{H}_{\text{scr}} := \mathcal{H}_{\text{scr}}/PGL(6).$$

We regard  $\mathfrak{H}_{\text{scr}}$  as the coarse moduli space of 3-nodal septic scrolls.

**Proposition 3.6.** *The parameter space  $\mathfrak{H}_{\text{scr}}$  is birationally isomorphic to  $\text{Sym}^3(\text{Sec}(R'))/\text{Aut}(R')$ . In particular,  $\mathfrak{H}_{\text{scr}}$  is a unirational 9-dimensional variety.*

*Proof.* We identify  $\text{Aut}(R')$  with the group consisting of linear automorphisms  $\sigma \in PGL(9)$  such that  $\sigma(R') = R'$ . Every  $\sigma \in \text{Aut}(R')$  clearly invariants  $\text{Sec}(R')$ . Since  $\text{Sing}(\text{Sec}(R')) = R'$ , conversely, every automorphism  $\sigma \in PGL(9)$  invariating  $\text{Sec}(R')$  belongs actually to  $\text{Aut}(R')$ . One has a birational action of  $\text{Aut}(R')$  on  $\text{Sym}^3(\text{Sec}(R'))$  given by

$$\sigma \langle a_1, a_2, a_3 \rangle := \langle \sigma(a_1), \sigma(a_2), \sigma(a_3) \rangle,$$

for  $\sigma \in \text{Aut}(R')$  and  $a_1, a_2, a_3 \in \text{Sec}(R')$ . We can now define a birational morphism

$$\vartheta : \text{Sym}^3(\text{Sec}(R'))/\text{Aut}(R') \dashrightarrow \mathfrak{H}_{\text{scr}}, \text{ by setting}$$

$$\Lambda := \langle a_1, a_2, a_3 \rangle \mapsto \pi_\Lambda(R') \bmod PGL(6),$$

where  $\pi_\Lambda : \mathbf{P}^9 \dashrightarrow \mathbf{P}^5$  is a projection of center  $\Lambda$ . The assignment is clearly  $\text{Aut}(R')$ -invariant, hence  $\vartheta$  is well-defined. Applying Lemma 3.1, we obtain that  $\vartheta$  is a birational isomorphism.

The secant variety  $\text{Sec}(R')$  being a  $\mathbf{P}^1$ -bundle over the rational variety  $\text{Sym}^2(R')$  is unirational. This implies that  $\text{Sym}^3(\text{Sec}(R'))$  and thus  $\mathfrak{H}_{\text{scr}}$  are unirational as well.  $\square$

Over the Hilbert scheme  $\mathcal{H}_{\text{scr}}$  we consider the universal family of scrolls:

$$\mathcal{H}_{\text{scr}} \xleftarrow{p_1} \mathcal{Y}_{\text{scr}} \xrightarrow{p_2} \mathbf{P}^5.$$

We introduce the incidence correspondence between cubic fourfolds of discriminant 26 and nodal septic scrolls in  $\mathbf{P}^5$ :

$$|\mathcal{O}_{\mathbf{P}^5}(3)| \longleftarrow \mathcal{X} := \mathbf{P}\left((p_1)_*(\mathcal{I}_{\mathcal{Y}_{\text{scr}}/\mathcal{H}_{\text{scr}} \times \mathbf{P}^5} \otimes p_2^* \mathcal{O}_{\mathbf{P}^5}(3))\right) \longrightarrow \mathcal{H}_{\text{scr}}$$

The group  $PGL(6)$  acts on the entire diagram. By quotienting out this action, if we set  $\mathfrak{X} := \mathcal{X}/PGL(6)$ , we obtain two projections:

$$\mathcal{C}_{26} \xleftarrow{\pi_1} \mathfrak{X} \xrightarrow{\pi_2} \mathfrak{H}_{\text{scr}}$$

The 21-dimensional variety  $\mathfrak{X}$  being a  $\mathbf{P}^{12}$ -bundle over the unirational variety  $\mathfrak{H}_{\text{scr}}$  is unirational as well. A general scroll  $[R] \in \mathfrak{H}_{\text{scr}}$  has precisely 3 non-normal nodes. Checking that a general cubic fourfold  $X \supseteq R$  is smooth, reduces to a standard Macaulay calculation. Applying (5), we obtain that the lattice  $A(X)$  contains a 2-dimensional lattice  $\langle h^2, [R] \rangle$  of discriminant 26, therefore the map  $\pi_1$  is well-defined. Proposition 2.1 implies  $\dim \pi_1^{-1}([X]) \leq 2$ , for all  $[X] \in \mathcal{C}_{26}$ , hence  $\mathfrak{X}$  dominates  $\mathcal{C}_{26}$ . In fact one can prove something more precise and establish in the process Theorem 1.2.

**Theorem 3.7.** *The incidence correspondence  $\mathfrak{X}$  is birational to the universal K3 surface  $\mathcal{F}_{14,1}$ .*

*Proof.* We define a map  $\theta : \mathfrak{X} \rightarrow \mathcal{F}_{14,1}$  as follows. We start with a pair  $[X, R] \in \mathfrak{X}$  and denote by  $f_R : \mathbf{P}^1 \rightarrow F(X)$  the rational curve of rulings described in Proposition 3.3. Denoting by  $[S, H] := \phi^{-1}([X]) \in \mathcal{F}_{14}$  the polarized K3 surface provided by the identification (3), applying Proposition 2.1, there exists a uniquely determined point  $p \in S$  such that  $\Delta_p = \Gamma_R$ .

The map  $\theta$  is clearly generically injective. Since both  $\mathfrak{X}$  and  $\mathcal{F}_{14,1}$  are irreducible varieties of the same dimension 21, it follows that  $\theta$  is birational. In particular, in the isomorphism  $S \cong \overline{\mathcal{M}}_7(X)$  constructed in Corollary 2.4, the general point on both sides corresponds to a septic scroll  $R \subseteq X$  which is 3-nodal and has no further singularities.  $\square$

#### 4. THE RATIONALITY OF $\mathcal{F}_{14,1}$

In this section, using in an essential way the characterization given in Proposition 3.3 of the rational curves  $\Gamma_R$  of rulings of 3-nodal scrolls  $R \subseteq \mathbf{P}^5$ , we show that the universal K3 surface of genus 14 is rational.

We begin by recalling the structure of the moduli space of curves of genus 8. Consider the Grassmannian  $\mathbf{G}(1, 5) \subseteq \mathbf{P}^{14}$  in its Plücker embedding. Denote by

$$\mathfrak{M}_8 := \mathbf{G}\left(7, \mathbf{P}\left(\bigwedge^2 \mathbb{C}^6\right)\right) / PGL(6)$$

the space of codimension 7 linear sections of  $\mathbf{G}(1, 5)$ . Mukai [M6] has shown that the map

$$\mathfrak{M}_8 \dashrightarrow \overline{\mathcal{M}}_8,$$

sending a general 7-plane  $[\mathbf{P}(V) \hookrightarrow \mathbf{P}^{14}] \in \mathfrak{M}_8$  to the intersection  $[\mathbf{G}(1, 5) \cdot \mathbf{P}(V)] \in \overline{\mathcal{M}}_8$  viewed as a canonical curve of genus 8, is a birational isomorphism. For more details on how to extend Mukai's isomorphism over parts of the boundary of  $\overline{\mathcal{M}}_8$ , see also [FV2].

Recall that we introduced in Section 3 the smooth septic scroll  $R' \cong \mathbf{F}_1 \subseteq \mathbf{P}^8$ , then considered a singular scroll  $R \subseteq \mathbf{P}^5$ , defined as the image of a linear projection  $\pi_\Lambda : R' \rightarrow \mathbf{P}^5$

whose center is a general plane  $\Lambda \subset \mathbf{P}^8$ , which is 3-secant to  $\text{Sec}(R')$ . We denote by  $p_1, p_2, p_3$  the three nodes of  $R$  and  $\{x_i, y_i\} = \pi^{-1}(p_i)$ . As explained in the Introduction,  $P_i \subseteq \mathbf{P}^5$  denotes the 2-plane spanned by the rulings of  $R$  passing through  $p_i$ , for  $i = 1, 2, 3$ . The line

$$L_i \subseteq \mathbf{G}(1, 5) \subseteq \mathbf{P}^{14}$$

parametrizes the lines in the plane  $P_i$  passing through the point  $p_i$ . If  $\Gamma = \Gamma_R \subseteq \mathbf{G}(1, 5)$  is the curve of rulings associated to  $R$  introduced in Proposition 3.3, then  $L_i$  meets  $\Gamma$  in two distinct points. We keep this notation throughout this section.

Due to the results of the previous section, our strategy is now to describe the family

$$\mathcal{U} \subseteq \text{Hom}(\mathbf{P}^1, \mathbf{G}(1, 5))$$

of smooth rational septic curves  $\Gamma_R \subseteq \mathbf{G}(1, 5)$  carrying three bisecant lines contained in  $\mathbf{G}(1, 5)$ . From Proposition 3.3 it follows that  $\mathcal{U}$  is birational to the Hilbert scheme  $\mathcal{H}_{\text{scr}}$  of 3-nodal septic scrolls in  $\mathbf{P}^5$ . Then we show that the quotient  $\mathcal{U}/PGL(6)$  is rational. Since  $\mathcal{U}/PGL(6)$  is birational to  $\mathfrak{H}_{\text{scr}}$  and, as proven in Theorem 1.2, the universal  $K3$  surface of genus 14 is a  $\mathbf{P}^{12}$ -bundle over  $\mathfrak{H}_{\text{scr}}$ , its rationality will follow.

The nodal curve  $\Gamma + L_1 + L_2 + L_3 \subseteq \langle \Gamma \rangle \cdot \mathbf{G}(1, 5)$  has arithmetic genus 3. It follows from Mukai's work [M1] that the intersection  $\langle \Gamma \rangle \cdot \mathbf{G}(1, 5)$  is a canonical curve of genus 8, provided (i) it is proper and reduced and (ii)  $\dim \langle \Gamma \rangle = 7$ . Using the surjectivity of the period map for polarized  $K3$  surfaces of genus 8, we shall show that both assumptions (i) and (ii) are satisfied. Granting both (i) and (ii) for the moment, we consider the canonically embedded curve in  $\langle \Gamma \rangle = \mathbf{P}^7$ , pictured also below:

$$(7) \quad C := \langle \Gamma \rangle \cdot \mathbf{G}(1, 5) = Q + \Gamma + L_1 + L_2 + L_3.$$

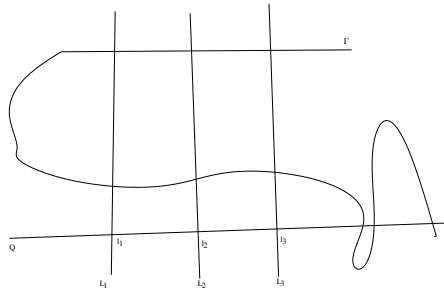


FIGURE 1. The canonical curve  $C = \Gamma + Q + L_1 + L_2 + L_3$ .

Bertini's Theorem implies that a general 8-dimensional space  $\langle \Gamma \rangle \subseteq \mathbf{P}^8 \subseteq \mathbf{P}^{14}$  cuts out on  $\mathbf{G}(1, 5)$  a smooth 2-dimensional linear section  $T$ , see also [Ve1], Propositions 3.2 and 3.3. By the adjunction formula,  $T \hookrightarrow \mathbf{P}^8$  is a smooth  $K3$  surface (of genus 8) polarized by  $\mathcal{O}_T(C)$ . We now describe the Picard lattice of  $T$ :

**Lemma 4.1.** *One has the following intersection products on  $T$ :*

$$Q^2 = -2, \quad Q \cdot \Gamma = 3, \quad Q \cdot L_i = 1, \quad \Gamma \cdot L_i = 2, \quad L_i \cdot L_j = -2\delta_{ij}, \quad \text{for } i, j = 1, 2, 3.$$

*Proof.* The generality assumptions ensure that  $L_i$  and  $L_j$  are disjoint lines, for  $i \neq j$ . Else, if  $L_i \cap L_j \neq \emptyset$ , then  $\langle p_i, p_j \rangle \subseteq P_i \cap P_j \subseteq \mathbf{P}^5$ . It follows that the four rulings of  $R'$  passing through the points  $x_i, y_i, x_j, y_j$  respectively, span a 6-dimensional space in  $\mathbf{P}^8$ , which is impossible for

$$h^0(R', \mathcal{O}_{R'}(1)(-4(\ell - E))) = h^0(R', \mathcal{O}_{R'}(E)) = 1,$$

where recall that  $\ell, E \in \text{Pic}(R')$  denote the line class and the exceptional divisor respectively. This implies that there exists a unique hyperplane in  $\mathbf{P}^8$  containing the four rulings, therefore they must span a 7-dimensional linear space.

Since  $L_i^2 = -2$ , by intersecting (7) with  $L_i$ , we obtain  $Q \cdot L_i = 1$ . Furthermore  $7 = \Gamma \cdot C$  and since  $\Gamma^2 = -2$ , we obtain  $\Gamma \cdot Q = 3$ . Finally,  $C \cdot Q = \deg(Q) = 4$ , therefore  $Q^2 + \Gamma \cdot Q + 3 = 4$ , implying  $Q^2 = -2$  and thus finishing the proof.  $\square$

In particular  $Q \subseteq \langle T \rangle = \mathbf{P}^8$  is a reduced, connected quartic curve of arithmetic genus zero. Since  $C - Q \equiv \Gamma + L_1 + L_2 + L_3$ , we obtain  $h^0(T, \mathcal{O}_T(C - Q)) = 4$ . The next lemma summarizes the situation.

**Lemma 4.2.** *The span  $\langle Q \rangle$  is 4-dimensional and  $Q$  is a connected nodal quartic curve with  $p_a(Q) = 0$ .*

In fact, we shall construct a  $K3$  surface  $T$ , such that the curve  $Q$  described in Lemma 4.2 is actually smooth.

To establish the validity of the assumptions (i) and (ii) and thus the existence of the special  $K3$  surface  $T$ , we use Hodge theory. We consider the following sublattice

$$(8) \quad \mathbb{L} := \mathbb{Z} \cdot [Q] \oplus \mathbb{Z} \cdot [\Gamma] \oplus \mathbb{Z} \cdot [L_1] \oplus \mathbb{Z} \cdot [L_2] \oplus \mathbb{Z} \cdot [L_3]$$

generated by the  $(-2)$  classes corresponding to  $Q, \Gamma, L_1, L_2$  and  $L_3$  respectively, and with intersection pairing as given in Lemma 4.1. We invoke the surjectivity of the period map for  $K3$  surfaces. The rank 5 lattice  $\mathbb{L}$  is even and has signature  $(1, 4)$ . Applying [Mo] Corollary 2.9, there exists a smooth  $K3$  surface  $T$ , such that  $\text{Pic}(T) \cong \mathbb{L}$ . We define the following class on  $T$

$$C := \Gamma + Q + L_1 + L_2 + L_3.$$

The genus zero curves  $\Gamma, Q, L_1, L_2, L_3 \subseteq T$  cannot have multiple components, for that would make  $\text{Pic}(T)$  larger than  $\mathbb{L}$ , therefore they are all smooth, rational curves on  $T$ .

**Lemma 4.3.** *The linear system  $|\mathcal{O}_T(C)|$  is very ample.*

*Proof.* We use Reider's Theorem [R], which, in the case of  $K3$  surfaces, had been proven before in [SD]. It suffices to show that there exists no curve  $E$  on  $T$  with  $E^2 = 0$  and  $E \cdot C \in \{1, 2\}$ , nor a curve  $F$  on  $T$  with  $F^2 = -2$  and  $F \cdot C = 0$ . We prove the first statement, the second follows similarly. Assuming there is such a curve  $E$ , we express it as an integral combination  $E \equiv x\Gamma + yQ + z_1L_1 + z_2L_2 + z_3L_3$  of the generators of  $\text{Pic}(T)$ . If  $C \cdot E = 1$ , we obtain

$$-15x^2 - 12xy - 5y^2 + 2x + y = z_1^2 + z_2^2 + z_3^2.$$

By comparing the signs of the two sides, one concludes that this equation has no integral solutions. The case  $C \cdot E = 2$  is similar. Finally, if  $F \equiv x\Gamma + yQ + z_1L_1 + z_2L_2 + z_3L_3$  is a  $(-2)$ -curve with  $C \cdot F = 0$ , we obtain

$$-15x^2 - 12xy - 5y^2 + 1 = z_1^2 + z_2^2 + z_3^2,$$

which implies  $x = y = 0$  and, say  $z_2 = z_3 = 0$  and then  $z_1 = 1$ . Thus  $F = L_1$ , but  $C \cdot L_1 = 1$ , hence this case does not appear. We conclude that  $C$  is very ample.  $\square$

We show that the  $K3$  surface  $T$  constructed in Lemma 4.3 is a linear section of  $\mathbf{G}(1, 5)$ . In particular, Mukai's results [M6] will apply for its hyperplane section  $C$ .

**Proposition 4.4.** *The  $K3$  surface  $T$  carries a globally generated rank two vector bundle  $T$  with  $\det(T) = \mathcal{O}_T(C)$ , providing an embedding  $T \hookrightarrow \mathbf{G}(1, 5)$  such that*

$$\langle T \rangle \cdot \mathbf{G}(1, 5) = S.$$

*Proof.* We use [M7] and need to show that the polarized  $K3$  surface  $(T, \mathcal{O}_T(C))$  is Brill-Noether general, that is, for all pairs of line bundles  $M, N$  on  $T$  such that  $M \otimes N = \mathcal{O}_T(C)$ , one has  $h^0(T, M) \cdot h^0(T, N) < h^0(T, C)$ . Under these circumstances, it is shown in *loc.cit.* that  $T$  carries a rigid, globally generated, stable rank 2 vector bundle  $E$  with  $h^0(T, E) = 6$  and  $\det(E) = \mathcal{O}_T(C)$ , inducing a map  $\varphi_E : T \rightarrow \mathbf{G}(1, 5)$ . Reasoning along the lines of [M7] Theorem 3.10, the  $K3$  surface  $T$  is then a linear section of  $\mathbf{G}(1, 5)$  in its Plücker embedding, that is,  $T = \mathbf{G}(1, 5) \cdot \langle T \rangle$ .

To establish the Brill-Noether generality of  $(T, \mathcal{O}_T(C))$ , we use for instance [GLT] Lemma 2.8. It suffices to show that in the lattice  $\mathbb{L}$  there exists no vector  $D$  such that  $D^2 = 2$  and  $D \cdot C \in \{7, 6\}$ , nor is there a vector  $D$  with  $D^2 = 0$  and  $D \cdot C \leq 4$ .

We treat in detail only the first case, the remaining ones being similar. We write

$$D \equiv x\Gamma + yQ + z_1L_1 + z_2L_2 + z_3L_3.$$

The conditions  $D^2 = 2$  and  $D \cdot C = 7$  translate into the equalities  $z_1 + z_2 + z_3 + 7x + 4y = 7$  and  $-15x^2 - 5y^2 - 12xy + 14x + 7y + 1 = z_1^2 + z_2^2 + z_3^2 \geq 0$ . It is elementary to see that there are no integral solutions.  $\square$

Using Proposition 4.4, we conclude that the intersection (7) corresponding to a general curve  $\Gamma_R \in \mathcal{U}$  corresponds to a semistable canonical curve of genus 8.

It will be useful to have a criterion for determining when the curve  $\Gamma$  spans a space of maximal possible dimension in the Plücker space  $\mathbf{P}^{14} \supseteq \mathbf{G}(1, 5)$ . To that end, recall that the Plücker embedding of the dual Grassmannian  $\mathbf{G}(1, 5)^\vee = \mathbf{G}(3, 5) \hookrightarrow (\mathbf{P}^{14})^\vee$  assigns to a point  $p \in \mathbf{G}(1, 5)^\vee$  corresponding to a 3-plane  $\mathbf{P}_p^3 \subseteq \mathbf{P}^5$  the Schubert cycle

$$\sigma_p := \{\ell \in \mathbf{G}(1, 5) : \ell \cap \mathbf{P}_p^3 \neq \emptyset\}.$$

Note that  $\dim \langle \Gamma \rangle + 1 = \text{codim} \langle \Gamma \rangle^\perp$ . Setting

$$W^1(\Gamma) := \mathbf{G}(3, 5) \cap \langle \Gamma \rangle^\perp = \{p \in \mathbf{G}(3, 5) : \Gamma \subseteq \sigma_p\},$$

for dimension reasons, the next lemma follows immediately:

**Lemma 4.5.** *Assume  $W^1(\Gamma)$  is finite. Then  $\dim \langle \Gamma \rangle = 7$ .*

Keeping the previous notation, let  $f_R : \mathbf{P}^1 \rightarrow \mathbf{G}(1, 5)$  be a sufficiently general element of  $\mathcal{U}$  and set again  $\Gamma = \Gamma_R$ . Then under the assumption  $R' = S_{3,4}$ , we can prove that:

**Theorem 4.6.** *The set  $W^1(\Gamma)$  is finite. In particular  $\dim \langle \Gamma \rangle = 7$  and  $\Gamma$  is a rational normal septic curve.*

*Proof.* If  $p \in W^1(\Gamma)$ , then  $\mathbf{P}_p^3$  contains an integral curve intersecting each line of  $R$ . Its strict transform by  $\pi_\Lambda : R' \rightarrow R$  is an integral section  $A$  of the ruled surface  $R'$ . Set  $d := \deg(A)$ , hence  $A \equiv (d-3)\ell - (d-4)E \in \text{Pic}(\mathbf{F}_1)$ . Clearly  $\langle A \rangle \subseteq \pi_\Lambda^{-1}(\mathbf{P}_p^3)$ , implying  $\dim \langle A \rangle \leq 6$ .

Let  $\mathbf{I}_A := |H - A|$  be the linear system of hyperplanes in  $\mathbf{P}^8$  containing the curve  $A \subseteq R'$ . By direct calculation, we find  $\dim(\mathbf{I}_A) = \dim |H - A| = 7 - d \geq 1$  and  $\dim |A| = 2d - 6$ . It follows that  $3 \leq d \leq 6$ . Recalling that  $V = H^0(\mathbf{P}^8, \mathcal{I}_{\Lambda/\mathbf{P}^8}(1))$ , the condition

$$\dim(\mathbf{P}V \cap \mathbf{I}_A) \geq 1$$

is equivalent to the condition that the curve  $\pi_\Lambda(A)$  be contained in a 3-space  $\mathbf{P}_p^3$ . For  $3 \leq d \leq 6$  let  $\mathbf{G}(7 - d, |H|)$  denote the Grassmannian of  $(7 - d)$ -subspaces of  $|H| \cong \mathbf{P}^8$  and introduce the  $(2d - 6)$ -dimensional variety

$$\mathbf{S}_d := \left\{ \mathbf{I}_{A'} \in \mathbf{G}(7 - d, |H|) : A' \in |(d - 3)\ell - (d - 4)E| \right\}.$$

For an integer  $k \geq 1$ , we consider the Schubert cycle

$$\sigma_V^k := \left\{ \mathbf{I} \in \mathbf{G}(7 - d, |H|) : \dim(\mathbf{P}V \cap \mathbf{I}) \geq k \right\}.$$

The cycle  $\sigma_V^k \cdot \mathbf{S}_d$  is finite for  $k = 1$  and empty for  $k \geq 2$ , provided the intersection is proper. By Kleiman's transversality of a general translate this is true for a general translate of  $\sigma_V^k$  in  $\mathbf{G}(d - 7, |H|)$ , that is, for a general choice of  $\Lambda$  (or equivalently, of  $V$ ). Hence  $W^1(\Gamma)$  is finite.  $\square$

**Remark 4.7.** The theorem above fails for rational septic scrolls in  $\mathbf{P}^8$  containing sections of degree  $d \leq 2$ , that is, for the scrolls  $S_{a,7-a}$ , where  $a \neq 3$ .

We turn to the smooth residual rational curve  $Q \subseteq \mathbf{G}(1, 5)$  defined by (7). Let

$$R_Q \subseteq \mathbf{P}^5$$

be the quartic scroll whose rulings are parametrized by the curve  $Q$ .

**Lemma 4.8.**  *$R_Q$  is a non-degenerate smooth rational normal scroll in  $\mathbf{P}^5$ .*

*Proof.* First, observe that  $R_Q$  cannot be a cone. Let us assume  $R_Q$  is a cone of vertex  $v \in \mathbf{P}^5$ . Then  $\langle Q \rangle \cong \mathbf{P}^4 \subseteq \mathbf{G}(1, 5)$  parametrizes the lines passing through  $v$ . This is a contradiction because  $\langle Q \rangle \subseteq \langle \Gamma \rangle \cdot \mathbf{G}(1, 5) = C$ . Now assume that  $R_Q$  is contained in a hyperplane  $H \subseteq \mathbf{P}^5$ . Then  $Q$  is contained in the Grassmannian  $\mathbf{G}_H := \mathbf{G}(1, H) \subseteq \mathbf{G}(1, 5)$  of lines of  $H$ . Since  $K_{\mathbf{G}_H} = \mathcal{O}_{\mathbf{G}_H}(-5)$ , we observe that, by adjunction, the curvilinear sections of  $\mathbf{G}_H$  are curves of arithmetic genus 1. Because of this fact and since  $\deg(\mathbf{G}_H) = 5$ , it follows that

$$\langle Q \rangle \cdot \mathbf{G}_H = Q + L \subseteq C,$$

where  $L$  is a bisecant line to  $Q$ . But the only line components in  $C$  are  $L_1, L_2, L_3$  and none of them is bisecant to  $Q$ . Via Proposition 2.3, the same argument shows that the scroll  $R_Q$  has no incident rulings, therefore  $R_Q$  is smooth.  $\square$

**Lemma 4.9.** *The scroll  $R_Q$  contains no other lines except the ruling parametrized by  $Q$ .*

*Proof.* Assume  $R_Q$  contains a line  $\ell_0$  not parametrized by a point of  $Q$ . We prove that this implies that  $W^1(\Gamma)$  is not finite, thus contradicting Theorem 4.6. Consider the family  $G$  of codimension 1 Schubert cycles  $\sigma_p$  defined by a 3-space  $\mathbf{P}_p^3 \supseteq \ell_0$ . Note that  $G \cong \mathbf{G}(1, 3)$ . We have  $G \subseteq \langle Q \rangle^\perp$ . Since  $\langle Q \rangle \subseteq \langle \Gamma \rangle$ , we also have  $\langle \Gamma \rangle^\perp \subseteq \langle Q \rangle^\perp$ . Counting dimensions it follows  $\dim(G \cap \langle \Gamma \rangle^\perp) \geq 1$ , which implies that  $W^1(\Gamma)$  is not finite.  $\square$

There are two types of smooth quartic scrolls in  $\mathbf{P}^5$ , namely  $S_{1,3} = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(3))$  and  $S_{2,2} = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(2))$ . The latter case is characterized by the property that every line contained in the scroll is a ruling. Lemma 4.9 implies the following:

**Theorem 4.10.** *Let  $\Gamma \subseteq \mathbf{G}(1, 5)$  be a smooth septic rational curve corresponding to a general element of  $\mathcal{U}$  and  $Q \subseteq \mathbf{G}(1, 5)$  the residual quartic curve. Then  $R_Q$  is isomorphic to  $S_{2,2}$ .*

To summarize, to a general rational curve  $\Gamma = \Gamma_R \in \mathcal{U}$ , we associated the quartic scroll  $R_Q$ , equipped with three rulings  $\ell_1, \ell_2, \ell_3$  corresponding to the points  $L_i \cdot Q \in \mathbf{G}(1, 5)$ , for  $i = 1, 2, 3$ . Each ruling  $\ell_i$  passes through the node  $p_i$  of the scroll  $R$  and is contained in the 2-plane  $P_i$  whose existence is established in Proposition 3.3.

To prove the rationality of  $\mathfrak{H}_{\text{scr}}$  and thus that  $\mathcal{F}_{14,1}$ , we reverse this construction. We denote by  $\mathcal{V}$  the variety classifying elements  $(R_Q, p_1, p_2, p_3)$ , where  $R_Q \subseteq \mathbf{P}^5$  is a smooth quartic scroll isomorphic to  $S_{2,2}$  and  $p_i \in R_Q$  for  $i = 1, 2, 3$ .

**Lemma 4.11.** *The  $PGL(6)$ -stabilizer of a general point  $(R_Q, p_1, p_2, p_3) \in \mathcal{V}$  is trivial. In particular,  $PGL(6)$  acts transitively on  $\mathcal{V}$ .*

*Proof.* The automorphism group of  $S_{2,2} \cong \mathbf{F}_0$  is the semidirect product of  $PGL(2) \times PGL(2)$  with  $\mathbb{Z}/2\mathbb{Z}$ . The last factor corresponds to the automorphism  $u \in \text{Aut}(\mathbf{F}_0)$  permuting the two factors. In particular,  $\text{Aut}(S_{2,2})$  is 6-dimensional. This implies that the space  $\mathcal{V}$  has dimension

$$\dim PGL(6) - \dim \text{Aut}(S_{2,2}) + 3\dim(R_Q) = 35 = \dim PGL(6).$$

Choose general points  $p_i = (a_i, b_i) \in \mathbf{F}_0 \cong S_{2,2}$ , with  $a_i \neq b_i$ , for  $i = 1, 2, 3$ . Up to the action of  $u \in \text{Aut}(\mathbf{F}_0)$ , the stabilizer  $\text{Stab}_{PGL(6)}(R_Q, p_1, p_2, p_3)$  corresponds to pairs of automorphism  $(\sigma_1, \sigma_2) \in PGL(2) \times PGL(2)$ , such that  $\sigma_1(a_i) = a_i$  and  $\sigma_2(b_i) = b_i$ . Thus  $\sigma_1 = \sigma_2 = 1$ . The points  $p_i$  not lying on the diagonal of  $\mathbf{F}_0$ , the automorphism  $u$  does not fix any of them, thus the stabilizer in question is trivial. Since  $\mathcal{V}$  and  $PGL(6)$  have the same dimension, this also implies the transitivity of the  $PGL(6)$ -action on  $\mathcal{V}$ , as claimed.  $\square$

We can thus start by fixing once and for all the quartic scroll  $R_Q$ . Precisely, we embed the surface  $\mathbf{F}_0 := \mathbf{P}^1 \times \mathbf{P}^1$  in  $\mathbf{P}^5$  via the linear system  $|\mathcal{O}_{\mathbf{F}_0}(1, 2)|$  and denote by

$$R_0 \subseteq \mathbf{P}^5$$

the image quartic scroll. The rulings on  $R_0$  are the elements of the linear system  $|\mathcal{O}_{\mathbf{F}_0}(0, 1)|$ . Let  $Q_0 \subseteq \mathbf{G}(1, 5)$  be the curve of rulings of  $R_0$ . We then fix three points in  $\mathbf{F}_0$ , for instance

$$\sigma_1 := ([1 : 0], [0 : 1]), \sigma_2 := ([0 : 1], [1 : 0]) \text{ and } \sigma_3 := ([1 : 1], [-1 : -1]),$$

which we identify with their images in  $R_0$ . As explained in Lemma 4.11, the stabilizer subgroup  $G$  of  $PGL(6)$  fixing both  $R_0$  as well as the set  $\{\sigma_1, \sigma_2, \sigma_3\}$  is isomorphic to the subgroup of  $PGL(2) \times PGL(2)$  fixing the set  $\{\sigma_1, \sigma_2, \sigma_3\}$ . Therefore  $G = \mathfrak{S}_3$ .

For  $i = 1, 2, 3$ , we denote by  $\ell_i$  the ruling of  $R_0$  passing through the point  $\sigma_i$ . Then, let  $\mathbf{P}_i^3$  be the projective space consisting of 2-planes  $\Pi_i \subseteq \mathbf{P}^5$  containing the line  $\ell_i$ . Giving a plane  $\Pi_i$  is equivalent to specifying a line  $L_i \subseteq \mathbf{G}(1, 5)$  in the Plücker embedding of the Grassmannian. Note that  $L_i$  meets  $Q_0$  transversally at precisely one point, namely  $\ell_i \in \mathbf{G}(1, 5)$ .

We introduce a rational map

$$\varkappa : \mathbf{P}_1^3 \times \mathbf{P}_2^3 \times \mathbf{P}_3^3 / \mathfrak{S}_3 \dashrightarrow \mathfrak{H}_{\text{scr}}$$

defined as follows. To a triple of planes  $(\Pi_1, \Pi_2, \Pi_3)$ , we attach the lines  $L_1, L_2, L_3 \subseteq \mathbf{G}(1, 5)$ . Since  $Q_0 \subseteq \mathbf{G}(1, 5)$  is a smooth rational quartic curve, in the Plücker embedding we have that  $\langle Q_0 \rangle \cong \mathbf{P}^4$ . Attaching one general 1-secant line to  $Q_0$  increases the dimension of the linear span of the union by one, therefore by attaching three general 1-secant lines, we have

$$\langle Q_0 + L_1 + L_2 + L_3 \rangle \cong \mathbf{P}^7 \subseteq \mathbf{P}^{14}.$$



We write

$$\langle Q_0 + L_1 + L_2 + L_3 \rangle \cdot \mathbf{G}(1, 5) = Q_0 + L_1 + L_2 + L_3 + \Gamma,$$

where  $\Gamma$  is a degree 7 curve. Applying Lemma 4.1, it follows that  $\Gamma$  is a rational curve and  $\Gamma \cdot L_i = 2$ , for  $i = 1, 2, 3$ . We denote by  $\ell'_i$  and  $\ell''_i$  the intersection points  $L_i \cdot \Gamma$ . From Proposition 3.3 it follows that the scroll  $R := R_\Gamma$  induced by  $\Gamma$  is 3-nodal, with nodes given by the intersection  $\ell'_i \cap \ell''_i$  taken in the 2-plane  $\Pi_i$ . We set

$$\varkappa(\Pi_1 + \Pi_2 + \Pi_3) := [R].$$

We conclude the proof of the rationality of the Hilbert scheme of 3-nodal scrolls in  $\mathbf{P}^5$ :

*Proof of Theorem 1.3.* We first observe that  $\varkappa$  is well-defined. To that end, we choose the polarized  $K3$  surface  $(T, \mathcal{O}_T(C))$  constructed in Propositions 4.3 and 4.4 and we keep the notation used there. Applying Theorem 4.10, the residual quartic rational curve  $Q \subseteq \mathbf{G}(1, 5)$  parametrizes the rulings of a quartic scroll  $R_Q \subseteq \mathbf{P}^5$ , which is isomorphic to  $S_{2,2}$ . Applying Lemma 4.11, there exists a unique automorphisms  $\sigma \in PGL(6)$  such that  $\sigma(R_Q) = R_0$  and  $\sigma(p_i) = o_i$ , for  $i = 1, 2, 3$ . Set  $\sigma(P_i) =: \Pi_i \in \mathbf{P}^3$  and then  $\varkappa(\Pi_1 + \Pi_2 + \Pi_3) = [R_\Gamma]$ .

To finish the proof it suffices to observe that  $\varkappa$  is generically injective. A general septic curve  $\Gamma \in \mathcal{U}$  corresponding to a 3-nodal septic scroll  $[R_\Gamma] \in \mathfrak{H}_{\text{scr}}$  has precisely 3 bisecant lines lying in  $\mathbf{G}(1, 5)$ . Giving  $\Gamma$  determines its linear span  $\langle \Gamma \rangle$ , hence the set  $\{L_1, L_2, L_3\}$  as well.  $\square$

## 5. THE UNIRATIONALITY OF THE UNIVERSAL $K3$ SURFACE OF GENUS AT MOST 12

We denote by  $\mathcal{F}_{g,n}$  the universal  $n$ -pointed  $K3$  surface of genus  $g$ . Thus  $\mathcal{F}_{g,n}$  is an irreducible variety of dimension  $19 + 2n$ . Similarly, one can consider the universal Hilbert scheme of 0-dimensional cycles of length  $n$ , that is,  $u^{[n]} : \mathcal{F}_g^{[n]} \rightarrow \mathcal{F}_g$ . We also introduce the notation  $\mathcal{C}_{g,n} := \mathcal{M}_{g,n}/\mathfrak{S}_n$  for the degree  $n$  universal symmetric product over  $\mathcal{M}_g$ , where the symmetric group  $\mathfrak{S}_n$  acts by permuting the marked points.

The aim of this short last section is to point out how Mukai's results determine the birational type of  $\mathcal{F}_{g,n}$  and that of  $\mathcal{F}_g^{[n]}$  for small  $g$ , and thus put our Theorem 1.1 better into context:

**Theorem 5.1.** *The following results on the Kodaira dimension of  $\mathcal{F}_{g,n}$  hold:*

- (i)  $\mathcal{F}_{g,g+1}$  is unirational for  $g \leq 10$ .
- (ii)  $\mathcal{F}_{11,1}$  is unirational. The Kodaira dimension of both  $\mathcal{F}_{11,11}$  and  $\mathcal{F}_{11}^{[11]}$  equals 19.

*Proof.* For  $g \leq 5$ , the general  $K3$  surface of genus  $g$  is a complete intersection in a projective space and the result follows easily. For details, see the table after Theorem 1.10 in [M7].

For  $6 \leq g \leq 10$ , Mukai [M1] has constructed a rational homogeneous variety  $V_g \subseteq \mathbf{P}^{N_g}$ , where  $N_g = g + \dim(V_g) - 2$ , such that the general  $K3$  surface of genus  $g$  is obtained as a general linear section  $S = V_g \cap \Lambda_g$ , where  $\Lambda_g \subseteq \mathbf{P}^{N_g}$  is a  $g$ -dimensional plane, with the polarization being the one induced by  $\mathcal{O}_{\mathbf{P}^{N_g}}(1)$ . Moreover, one has the following birational isomorphism, see [M1] Corollary 0.3:

$$\mathcal{F}_g \xrightarrow{\cong} \mathbf{G}(g, N_g)/\text{Aut}(V_g).$$

These results imply the existence of a dominant map  $\chi_g : V_g^{g+1} \dashrightarrow \mathcal{F}_{g,g+1}$  given by

$$\chi(x_1, \dots, x_{g+1}) := [V_g \cap \langle x_1, \dots, x_{g+1} \rangle, x_1, \dots, x_{g+1}].$$

This proves that  $\mathcal{F}_{g,g+1}$  (and hence  $\mathcal{F}_{g,n}$  for  $n \leq g+1$ ) is unirational in this range.

For  $g = 11$ , we use [M8], where it is shown that a general curve  $[C] \in \mathcal{M}_{11}$  lies on a *unique*  $K3$  surface  $C \subseteq S$  as a hyperplane section, with  $\text{Pic}(S) = \mathbb{Z} \cdot C$ . This implies the existence of a rational map  $\chi_n : \mathcal{M}_{11,n} \dashrightarrow \mathcal{F}_{11,n}$  defined by

$$\chi_n([C, x_1, \dots, x_n]) := [S, x_1, \dots, x_n].$$

The map  $\chi_n$  is dominant for  $n \leq 11$  and a birational isomorphism for  $n = 11$ . Indeed, in this last case, given an embedded  $K3$  surface  $S \xrightarrow{|H|} \mathbf{P}^{11}$  and general points  $x_1, \dots, x_{11} \in S$ , the hyperplane  $\langle x_1, \dots, x_{11} \rangle \cong \mathbf{P}^{10}$  cuts out a canonical genus 11 curve  $C$  on  $S$ , which comes equipped with the marked points  $x_1, \dots, x_{11}$ . By quotienting the action of the symmetric group  $\mathfrak{S}_{11}$ , the map  $\chi_{11}$  induces a birational isomorphism between the universal symmetric product  $\mathcal{C}_{11,11}$  and  $\mathcal{F}_{11}^{[11]}$ . Now we use [FV1] Theorem 0.5. Both varieties  $\mathcal{M}_{11,11}$  and  $\mathcal{C}_{11,11}$  have Kodaira dimension 19, hence we conclude.

We now pass on to the universal  $K3$  surface  $\mathcal{F}_{11,1}$ . To that end we define a rational map

$$\vartheta : \mathcal{M}_{10,2} \dashrightarrow \mathcal{F}_{11,1},$$

associating to a 2-pointed curve  $[C, p_1, p_2] \in \mathcal{M}_{10,2}$ , the unique  $K3$  surface  $S$  of genus 11 containing the curve  $[X := C/p_1 \sim p_2]$  obtained from  $C$  by identifying  $p_1$  and  $p_2$ . To show that  $\vartheta$  is well-defined, that is, Mukai's construction [M8] can be also carried out for the 1-nodal curve  $[X] \in \overline{\mathcal{M}}_{11}$ , we use [CLM] Proposition 4.4. Observe that the  $K3$  surface  $S$  has a distinguished point corresponding to the image of the singularity of  $X$ . The map  $\vartheta$  is clearly dominant, for in each linear system on a  $K3$  surface, the 1-nodal curves fill-up a divisor. The unirationality of  $\mathcal{F}_{11,1}$  now follows from that of  $\mathcal{M}_{10,2}$ , which can be established in a variety of ways, see for instance [BCF] Theorem B.

□

**Remark 5.2.** It is claimed incorrectly in [L] Table 3, that  $\mathcal{M}_{11,n}$  is unirational for  $n \leq 10$ . The argument sketched in *loc.cit.* only establishes the uniruledness of  $\mathcal{M}_{11,n}$  when  $n \leq 10$ , precisely using the map  $\chi_n : \mathcal{M}_{11,n} \rightarrow \mathcal{F}_{11,n}$ , which is birationally a  $\mathbf{P}^{11-n}$ -bundle. But this argument alone offers no indications concerning the birational nature of the base variety  $\mathcal{F}_{11,n}$ . One can establish partial results on the birational nature of  $\mathcal{F}_{11,n}$ , for  $n \leq 10$ . For instance, it is shown in [Ve1] that the universal product  $\mathcal{C}_{11,6}$  is unirational, which implies that  $\mathcal{F}_{11}^{[6]}$  is unirational as well.

**Remark 5.3.** Mukai [M4] gives an explicit orbit space realization over a projective space for the universal  $K3$  surface  $\mathcal{F}_{13,1}$ . The unirationality of  $\mathcal{F}_{13,1}$  thus follows. Presumably, a similar argument works for genus 12, when  $\mathcal{F}_{12}$  is known to be birational to a  $\mathbf{P}^{13}$ -bundle over the rational moduli space  $\mathcal{MF}_{22}$  of Fano 3-folds  $V_{22} \subseteq \mathbf{P}^{13}$ , see again [M1].

**Remark 5.4.** Since  $u : \mathcal{F}_{g,1} \rightarrow \mathcal{F}_g$  is a morphism fibred in Calabi-Yau varieties, by Iitaka's easy addition formula  $\kappa(\mathcal{F}_{g,1}) \leq \dim(\mathcal{F}_g) = 19$ , in particular,  $\mathcal{F}_{g,1}$  is never of general type. Furthermore, by [K], we also write  $\kappa(\mathcal{F}_{g,1}) \geq \kappa(\mathcal{F}_g)$ . In particular, when  $\mathcal{F}_g$  is of general type, then  $\kappa(\mathcal{F}_{g,1}) = 19$ .

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